**Phys 410**
**Fall 2014**
**Lecture #29 Summary**
**11 December, 2014**

We concluded our discussion of Special Relativity. Einstein made two postulates:

1) If S is an inertial reference frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial reference frame.
2) The speed of light (in vacuum) has the same value $c$ in every direction in all inertial reference frames.

By comparing two forms of the four-momentum, namely $p^{(4)} = m\gamma(v)(\vec{v}, c)$ and $p^{(4)} = (\vec{p}, E/c)$, and taking the ratio of the 3-vector part to the fourth component, we derive an expression for the velocity in terms of the relativistic momentum and energy: $\vec{\beta} = \frac{\vec{v}}{c} = \vec{p}c/E$, or $\vec{v} = \vec{p}c^2/E$. Using the definition of relativistic energy $E = \gamma(v)mc^2$, we can also write $\vec{v} = \vec{p}/(\gamma(v)m)$, which clearly reduces to the classical expression in the limit $\frac{v}{c} \ll 1$.

Next, consider the four-momentum of a particle of mass $m$ at rest in your reference frame. In this case $p^{(4)} = (0,0,0,mc)$. The invariant length of this four-vector is $p^{(4)} \cdot p^{(4)} = -(mc)^2$. Now boost into another reference frame moving at speed $-v$. In that frame the particle is moving by at speed $v$ and has a four-momentum given by $p^{(4)} = (\vec{p}, E/c)$. It's invariant length has the same value as that for the observer in the particle's rest frame. Using this equality we can find the useful relation: $E^2 = (\vec{p}c)^2 + (mc^2)^2$. The total relativistic energy is not simply a sum of rest energy and kinetic energy, but these quantities are related like the sides of a right triangle!

We then considered Compton scattering as an application of the conservation of four-momentum. One fires x-rays of a given frequency (or wavelength) at stationary electrons. The x-rays scatter off at an angle $\theta$ with respect to the incident direction and change their frequency, and the electron is kicked away as well. We equated the total four-momentum before and after the collision. The photon (now treated as a particle rather than a wave) four-momentum can be constructed using two relations from quantum mechanics ($E = \hbar\omega$, and $\vec{p} = \hbar\vec{k}$). With this one can say for a photon $p_\gamma^{(4)} = \hbar\left(\vec{k}, \frac{\omega}{c}\right) = \frac{\hbar\omega}{c}(\hat{k}, 1)$. One then writes the sum of the photon and stationary electron four-momentum before the collision and equates it to the sum after the collision. Square both sides and take advantage of two scalar invariant results, namely that $p_\gamma^{(4)} \cdot p_\gamma^{(4)} = 0$, and for the electron $p_{initial}^{(4)} \cdot p_{initial}^{(4)} = p_{final}^{(4)} \cdot p_{final}^{(4)}$. With this one can derive the famous Compton scattering formula: $\lambda - \lambda_0 = \frac{h}{mc}(1 - \cos\theta)$.

Finally we considered the definition of a four-force as the derivative of the four-momentum with respect to the differential proper time interval: $K^{(4)} = \frac{dp^{(4)}}{dt_0} = \gamma\left(\frac{d\vec{p}}{dt}, \frac{1}{c}\frac{dE}{dt}\right)$. Define the relativistic three-force as $\vec{F} = d\vec{p}/dt$, where $\vec{p}$ is the relativistic three-momentum. By exploiting the useful expression derived above, the four-force can be written as: $K^{(4)} = \gamma\left(\vec{F}, \frac{1}{c}\vec{v}\cdot\vec{F}\right)$. We considered a massive particle subjected to a constant force (for all time), starting from rest at the origin. The relativistic momentum is easy to calculate: $\vec{p} = \vec{F}t$, and it would appear from this that the particle velocity should eventually equal and exceed the speed of light. However, if we calculate the velocity itself as $\vec{v} = \vec{p}/(\gamma(v)m)$, we find that it never reaches the speed of light (in finite time)! We find $\vec{v} = (\frac{\vec{F}t}{m})/\sqrt{1 + (Ft/mc)^2}$. Thus the speed of light is the ultimate speed limit of the universe.